

Principal models on a solvable group with nonconstant metric

LADISLAV HLAVATÝ *

Department of Physics,

Faculty of Nuclear Sciences and Physical Engineering

February 8, 2008

Abstract

Field equations for generalized principle models with nonconstant metric are derived and ansatz for their Lax pairs is given. Equations that define the Lax pairs are solved for the simplest solvable group. The solution is dependent on one free variable that can serve as the spectral parameter. Painlevé analysis of the resulting model is performed and its particular solutions are found

1991 MSC numbers: 35L10, 35L15, 34A55

Keywords: principal models, chiral models, sigma models, Lax pair, integrable models, solvable group.

1 Introduction

The Lax pair formulation for principal models with the trivial (δ_{ij}) metric was published in [1] and Lax formulation of related $O(N)$ -sigma

*Postal address: Břehová 7, 115 19 Prague 1, Czech Republic. E-mail: hlavaty@br.fjfi.cvut.cz

models in [2], [3]. The models with the nontrivial but constant metric on the group manifold were investigated e.g. in [4] – [5]. The goal of this paper is to present an example of the principal model with a nonconstant metric on the group manifold having the Lax pair. The models with the nonconstant metrics we are aware of (e.g. [6], [7]) are different from those investigated in this paper.

In the previous work [8] we have given several conditions on the metric for $SU(2)$ models but we were not succesful in finding an example of the metric that is not constant on the group manifold and admits the Lax formulation of field equations. Trying to find such an example, we have chosen much simpler group, namely the group of affine transformations of the real line – denoted here $Af(1)$. It is a nonabelian solvable two–dimensional Lie group that can be realized as the group of matrices

$$g = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \quad a > 0, \quad b \in \mathbf{R}. \quad (1)$$

It is diffeomorphic to \mathbf{R}^2 and one can choose the global coordinates θ_1, θ_2 on $Af(1)$ setting e.g. $a = \exp(\theta_1)$, $b = \theta_2$. The basis in the corresponding two–dimensional solvable Lie algebra can be chosen that the only nonvanishing structure coefficients are

$$c_{12}^2 = -c_{21}^2 = 1 \quad (2)$$

The coordinates of the left–invariant fields $J_\mu := g^{-1}\partial_\mu g$ in this basis are $(\partial_\mu\theta_1, e^{-\theta_1}\partial_\mu\theta_2)$.

2 Formulation of generalized principal chiral models

Principal chiral models are given by the action

$$I[g] = \int d^2x \eta^{\mu\nu} L(J_\mu, J_\nu), \quad (3)$$

where

$$J_\mu := (g^{-1}\partial_\mu g) \in \mathcal{L}(G), \quad (4)$$

$g : \mathbf{R}^2 \rightarrow G$, $\mu, \nu \in \{0, 1\}$, $\eta := \text{diag}(1, -1)$ and L is the Killing form on the corresponding Lie algebra $\mathcal{L}(G)$.

An immediate generalization of the principal chiral models is obtained when one considers a general invertible bilinear form instead of the Killing [9],[5]. Next step of generalization (3) is introducing G -dependent symmetric bilinear forms. The generalized principal chiral models are then defined by the action

$$I[g] = \int d^2x L_{ab}(g) \eta^{\mu\nu} (g^{-1} \partial_\mu g)^a (g^{-1} \partial_\nu g)^b, \quad (5)$$

where $L_{ab}(g)$ is invertible symmetric matrix $\dim G \times \dim G$ defined by the G -dependent bilinear form $L(g)$ as

$$L_{ab}(g) := L(g)(t_a \otimes t_b), \quad (6)$$

and t_j are elements of a basis in the Lie algebra of the left-invariant fields. It is useful to consider the bilinear form $L(g)$ as a metric on the group manifold. Lie products of elements of the basis define the structure coefficients

$$[t_a, t_b] = c_{ab}^c t_c \quad (7)$$

and in the same basis we define the coordinates of the field J_ν

$$J_\nu = g^{-1} \partial_\nu g = J_\nu^b t_b \quad (8)$$

that satisfy the Bianchi identities

$$\partial_\mu J_\nu - \partial_\nu J_\mu + [J_\mu, J_\nu] = 0 \quad (9)$$

Varying the action (5) w.r.t. $\eta := g^{-1} \delta g$ we obtain equations of motion for the generalized principal chiral models

$$\partial_\mu J^{\mu,a} + \Gamma_{bc}^a J_\mu^b J^{\mu,c} = 0, \quad (10)$$

where

$$\Gamma_{bc}^a := S_{bc}^a + \gamma_{bc}^a, \quad (11)$$

S_{bc}^a is the so called flat connection given by the structure coefficients

$$S_{bc}^a := \frac{1}{2}(C_{bc}^a + C_{cb}^a), \quad C_{bc}^a := (L^{-1})^{ap} c_{pb}^q L_{qc} \quad (12)$$

and γ_{bc}^a are the Christoffel symbols for the metric $L_{ab}(g)$ on the group manifold

$$\gamma_{bc}^a := \frac{1}{2}(L^{-1})^{ad}(U_b L_{cd} + U_c L_{bd} - U_d L_{bc}). \quad (13)$$

The vector fields U_a in (13) are defined in the local group coordinates θ_i as

$$U_a := U_a^i(\theta) \frac{\partial}{\partial \theta_i} \quad (14)$$

where the matrix U is inverse to the matrix V of vielbein coordinates

$$U_a^i(\theta) := (V^{-1})_a^i(\theta), \quad V_i^a(\theta) := (g^{-1} \frac{\partial g}{\partial \theta_i})^a. \quad (15)$$

Note that the connection (11) is symmetric in the lower indices

$$\Gamma_{bc}^a = \Gamma_{cb}^a. \quad (16)$$

For the group $Af(1)$

$$U_a = (\frac{\partial}{\partial \theta_1}, e^{\theta_1} \frac{\partial}{\partial \theta_2}). \quad (17)$$

3 The Lax pairs

In the paper [5], the ansatz for the Lax formulation of the generalized chiral models was taken in the form

$$[\partial_0 + P_{ab} J_0^b t_a + Q_{ab} J_1^b t_a, \partial_1 + P_{ab} J_1^b t_a + Q_{ab} J_0^b t_a] = 0 \quad (18)$$

where P, Q are two auxiliary $\dim G \times \dim G$ matrices. The ansatz (18) is a generalization of the Lax pair for Killing metric on the compact semisimple group $L_{ab} = \text{Tr}(t_a t_b)$ (in that case Q and P are multiples of the unit matrix).

Necessary conditions that the operators in (18) form the Lax pair for the equations of motion (10) are

$$\partial_0 P = \partial_1 Q, \quad \partial_0 Q = \partial_1 P \quad (19)$$

$$(P_{bp} P_{cq} - Q_{bp} Q_{cq}) c_{bc}^a = P_{ab} c_{pq}^b, \quad (20)$$

$$\frac{1}{2} c_{cd}^a (P_{cp} Q_{dq} + P_{cq} Q_{dp}) = Q_{ab} \Gamma_{pq}^b. \quad (21)$$

If Q is invertible, that we shall assume in the following, then these conditions are *also sufficient*. Note that the first two conditions are independent of the bilinear form L so that one can start with solving the equation (20) and then look for the bilinear forms L that admit

solution of the equation (19,21). Moreover, as we have no *a priori* conditions for the first derivatives of the fields g i.e. derivatives of θ_1, θ_2 , we get from (19)

$$\partial_{\theta_j} P = 0, \quad \partial_{\theta_j} Q = 0 \quad (22)$$

3.1 Solution of the equations (19–21) for $Af(1)$

The equations (20–21) where $a = 1$ become linear equations for elements P_{1b} and Q_{1b} due to the fact that $c_{pq}^1 = 0$. It is rather easy to see that there is only trivial solution $Q_{11} = Q_{12} = 0$ of the equations $Q_{1b}\Gamma_{pq}^b = 0$ for constant nondegenerate metric L (i.e. $\Gamma_{pq}^b = S_{pq}^b$) so that *there is no invertible matrix Q that satisfy the equations (21) for $Af(1)$ and the constant nondegenerate metric L .*

On the other hand, there are solutions of the equations (20– 21) with $\det Q \neq 0$ for the nonconstant metric. In the following we shall present solutions for the diagonal metric

$$L(g) = \begin{pmatrix} k_1(\theta_1, \theta_2) & 0 \\ 0 & k_2(\theta_1, \theta_2) \end{pmatrix}. \quad (23)$$

The equation (20) for $Af(1)$ imply that matrix P is of the form

$$P = \begin{pmatrix} p_1 & 0 \\ p_3 & p_2 \end{pmatrix} \quad (24)$$

where

$$p_2(p_1 + 1) = \det Q \quad (25)$$

The linear equations for the elements Q_{1b} given by (21) read

$$\begin{aligned} Q_{1b}\gamma_{11}^b &= 0, \\ 2Q_{1b}\gamma_{12}^b + Q_{12} &= 0, \\ k_1Q_{1b}\gamma_{22}^b - k_2Q_{11} &= 0. \end{aligned} \quad (26)$$

The condition $\det Q \neq 0$ requires that this system has a nontrivial solution. The nontrivial solvability of this system implies a system of quadratic equations for the Christoffel symbols that finally give a system of partial differential equations for the elements of the metric

$$(k_2 - k_{2,1})k_{1,1} - e^{2\theta_1}k_{1,2}^2 = 0 \quad (27)$$

$$(2k_2 - k_{2,1})k_{1,2} + k_{1,1}k_{2,2} = 0 \quad (28)$$

$$(k_2 - k_{2,1})(2k_2 - k_{2,1}) + e^{2\theta_1}k_{1,2}k_{2,2} = 0 \quad (29)$$

where $k_{i,j} \equiv \frac{\partial k_i}{\partial \theta_j}$. For every solution of this system we can get a solution of the equations (20, 21) with $\det Q \neq 0$. However, the equations (19) or (22) can be satisfied only if

$$k_1 = K_1, \quad k_2 = K_2 \exp(2\theta_1), \quad (30)$$

or

$$k_1 = K_1 \exp(\theta_1), \quad k_2 = K_2 \exp(\theta_1), \quad (31)$$

where K_1, K_2 are arbitrary nonzero constants.

For the case (30), i.e. for

$$L(g) = \begin{pmatrix} K_1 & 0 \\ 0 & K_2 e^{2\theta_1} \end{pmatrix}, \quad (32)$$

we get

$$P = \begin{pmatrix} 1 + \epsilon q_1 & 0 \\ c_1(1 + \epsilon q_1)/q_1 & p_2 \end{pmatrix}, \quad Q = \begin{pmatrix} q_1 & 0 \\ c_1 & \epsilon p_2 \end{pmatrix}. \quad (33)$$

where $\epsilon = \pm 1$, c_1 , q_1 , p_2 are arbitrary constants $q_1 \neq 0$, $p_2 \neq 0$. The equations of motion of this model read

$$\partial_\nu J^{1\nu} = 0 \quad (34)$$

$$\partial_\nu J^{2\nu} + J_\nu^1 J^{2\nu} = 0 \quad (35)$$

that actually *represent the free model*

$$\partial_\nu \partial^\nu \theta_1 = 0 \quad (36)$$

$$\partial_\nu \partial^\nu \theta_2 = 0 \quad (37)$$

For the case (31) i.e. for

$$L(g) = \begin{pmatrix} K_1 e^{\theta_1} & 0 \\ 0 & K_2 e^{\theta_1} \end{pmatrix}, \quad (38)$$

the Lax pair is given by

$$P = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{q_2}{\kappa} & c_1 \kappa \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & \frac{\kappa}{2} \\ c_1 & q_2 \end{pmatrix} \quad (39)$$

where $\kappa^2 = -K_2/K_1 \neq 0$ and c_1, q_2 are constants, $c_1 \neq 0$. The equations of motion of this *nontrivial Af(1) model with the nonconstant metrics* are

$$\partial_\nu J^{1\nu} + \frac{1}{2}(J_\nu^1 J^{1\nu} - \kappa^2 J_\nu^2 J^{2\nu}) = 0 \quad (40)$$

$$\partial_\nu J^{2\nu} = 0 \quad (41)$$

or

$$\partial_\nu \partial^\nu \theta_1 + \frac{1}{2}(\partial_\nu \theta_1)(\partial^\nu \theta_1) - \frac{1}{2}\kappa^2 e^{-2\theta_1}(\partial_\nu \theta_2)(\partial^\nu \theta_2) = 0 \quad (42)$$

$$\partial_\nu \partial^\nu \theta_2 - (\partial_\nu \theta_1)(\partial^\nu \theta_2) = 0 \quad (43)$$

Inserting (39) into (18) we get a linear combination of the equations (40), (41) containing the parameter q_2 and we find that without loss of generality we can set $q_2 = 0$. The Lax operators for the system (40), (41) then read

$$L_0 = \partial_0 + \begin{pmatrix} \frac{1}{2}(J_0^1 + \kappa J_1^2) & \lambda(J_1^1 + \kappa J_0^2) \\ 0 & 0 \end{pmatrix} \quad (44)$$

$$L_1 = \partial_1 + \begin{pmatrix} \frac{1}{2}(J_1^1 + \kappa J_0^2) & \lambda(J_0^1 + \kappa J_1^2) \\ 0 & 0 \end{pmatrix} \quad (45)$$

where $\lambda = c_1$ is free (spectral) parameter.

4 Painlevé analysis and particular solutions

To check the integrability of the model (42),(43) we can apply the usual Painlevé test [10], [11] i.e. to check the existence of the generic solution in the form of series with appropriate number of undetermined coefficients.

To be able to perform the test we must first convert the equations (42),(43) to the polynomial form. It can be done by the substitution $Z = \exp(\theta_1/2)$, $W = \kappa\theta_2/2$ that transform the equations to the form

$$Z^3 \partial_\nu \partial^\nu Z - (\partial_\nu W)(\partial^\nu W) = 0 \quad (46)$$

$$Z \partial_\nu \partial^\nu W - 2(\partial_\nu Z)(\partial^\nu W) = 0 \quad (47)$$

The singular point analysis is trivial as all possible leading powers for the above system are nonnegative. The analysis around $Z = 0$ can be done by setting $Z = 1/Y$ and consequent singular point analysis of the system

$$Y \partial_\nu \partial^\nu Y - 2(\partial_\nu Y)(\partial^\nu Y) + Y^6(\partial_\nu W)(\partial^\nu W) = 0 \quad (48)$$

$$Y \partial_\nu \partial^\nu W + 2(\partial_\nu Y)(\partial^\nu W) = 0 \quad (49)$$

The singular solutions of this system have leading powers of Y and W $(-1,3)$ and $(-1,0)$. The resonances are $r = -3, -1, 0, 0$ and $r = -1, 0, 0, 3$ respectively and compatibility conditions are satisfied. (In the former case they are trivial and in the latter case we have used a Mathematica code for the check.)

The polynomial form (46), (47) of the field equations is also convenient for obtaining particular solutions of the model. Indeed, choosing $Z(x_0, x_1) = z(y)$, $W(x_0, x_1) = w(y)$ where $y = f(x_0, x_1)$ and f solving the wave equation $\partial_\nu \partial^\nu f = 0$ we obtain from (46), (47) a system of ODEs for z, w that is rather easy to solve. We get

$$z(y) = Ae^{By} + Ce^{-By},$$

$$w(y) = D \pm \frac{1}{2}(A^2 e^{2By} + 4ABCy - C^2 e^{-2By})$$

or

$$z(y) = y, \quad w(y) = C$$

where A, B, C, D are arbitrary real constants.

5 Conclusion

We have classified the generalized principal $Af(1)$ models that possess the Lax pair of the form (18). We have found that no such model with constant metric L_{ab} in the action (5) exists and that there are just two models with the nonconstant diagonal metric. One of them is the two-component free model and the other one is given by the equations of motion (42), (43). A particular solution of the latter model was obtained and the Painlevé analysis confirms its integrability.

References

- [1] V.E. Zakharov and A.V. Mikhailov. Relativistically invariant two-dimensional models in field theory integrable by the inverse problem technique. *J.Exp.Theor.Phys.*, 74(6):1953–1973, 1978. In Russian.
- [2] K. Pohlmayer. Integrable hamiltonian systems and interaction through quadratic constraints. *Comm. Math. Phys.*, 46:207–221, 1976.
- [3] F. Lund. Example of a relativistic, completely integrable, hamiltonian systems. *Phys.Rev.Lett.*, 38:1175–1178, 1977.
- [4] I.V. Cherednik. Relativistically invariant quasiclassical limits of integrable two-dimensional quantum models. *Theor.Math.Phys.*, 47(2):225–229, 1981. In Russian.
- [5] N. Sochen. Integrable generalized principal chiral models. *Phys.Lett. B*, 391:374–380, 1997.
- [6] D. Maison. Are the stationary, axially symmetric Einstein equations completely integrable? *Phys. Rev. Lett*, 41(8):521–522, 1978.
- [7] M. Gürses and A. Karasu. Integrable sigma models. *Int.J.Mod.Phys. A*, 6(3):1487–500, 1991.
- [8] L. Hlavatý. Towards the Lax formulation of $SU(2)$ principal models with nonconstant metric. *e-print solv-int/9908008*.
- [9] I.V. Cherednik. On the integrability of the equation for the two-dimensional chiral asymmetric $O(3)$ field and its quantum analog. *Jadernaja fizika*, 33(1):278–282, 1981. In Russian.
- [10] M.J. Ablowitz, A. Ramani, and H. Segur. A connection between nonlinear evolution equations and ordinary differential equations of P-type. *J.Math.Phys.*, 21:715, 1980.
- [11] J. Weiss, M. Tabor, and G. Carnevale. The Painlevé property for partial differential equations. *J.Math.Phys.*, 24:522, 1983.